THE INTRINSIC EQUATION OF A CURVE IN POLAR CO-ORDINATES.

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The intrinsic equation of a curve has never been treated in polar co-ordinates in works on the subject. It is the aim of this paper to treat this subject briefly and to show that this method is somewhat simpler than the method of rectangular co-ordinates.

Let S denote the length of an arc of the curve $r=f(\theta)$ measured from some fixed point, φ the inclination of the tangent at the variable extremity to the tangent at the fixed point.

Let $y=f(x)$, then $\frac{dy}{dx}=f'(x)=-\cot\varphi \quad x=r\cos\theta, \quad y=r\sin\theta$

$\frac{dx}{d\theta}=\cos\theta\frac{dr}{d\theta}-r\sin\theta, \frac{dy}{d\theta}=\sin\theta\frac{dr}{d\theta}+r\cos\theta.$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\sin\theta\frac{dr}{d\theta}+r\cos\theta}{\cos\theta\frac{dr}{d\theta}-r\sin\theta}=-\cos\varphi.$$

$\therefore r\sin\theta\cot\varphi-\cos\theta\cot\varphi\frac{dr}{d\theta}=\sin\theta\frac{dr}{d\theta}+r\cos\theta.$

$$\therefore \frac{dr}{d\varphi}=r\tan(\theta-\varphi)=f'(\theta)$$

$\therefore f'(\theta)=f(\theta)\tan(\theta-\varphi)\ldots\ldots A.$

From A, θ is known in terms of φ, say $F(\varphi)$.

Then $\frac{d\theta}{d\varphi}=F'(\varphi)$.

$$\text{Now } \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}=\left[\left\{f(\theta)\right\}^2+\left\{f'(\theta)\right\}^2\right]^{\frac{1}{2}}$$

$\therefore \frac{ds}{d\theta}=\left\{r^2+r^2\tan^2(\theta-\varphi)\right\}^{\frac{1}{2}}=r\sec(\theta-\varphi)=f(\theta)\sec(\theta-\varphi).$

$$\frac{ds}{d\varphi}=\frac{ds}{d\theta}\cdot\frac{d\theta}{d\varphi}=F'(\varphi)f(\theta)\sec(\theta-\varphi)=rF'(\varphi)\sec(\theta-\varphi)\ldots\ldots(B).$$

From B by substituting the value of r in terms of θ, and the value of θ, as found in A, in terms of φ, S may be found in terms of φ by integration.

I. Required the intrinsic equation of the circle.

The polar equation is $r=a$.

$\therefore \frac{dr}{d\theta}=0=r\tan(\theta-\varphi). \quad \therefore \theta-\varphi=0, \text{ or } \theta=\varphi\ldots\ldots(1).$

$$\frac{d\theta}{d\varphi}=1, \therefore \frac{ds}{d\varphi}=a\sec(\varphi-\varphi)=a\sec 0=a. \quad \therefore S=a\varphi.$$

Or we may have proceeded thus:

$$\frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}=a,\quad \therefore S=a\theta, \text{ but } \theta=\varphi,\quad \therefore S=a\varphi.$$

Taking the equation $r=2a\cos\theta\ \frac{dr}{d\theta}=-2a\sin\theta=r\tan(\theta=\varphi)$.

$$\therefore -2a\sin\theta=2a\cos\theta\tan(\theta-\varphi).$$

$$\therefore -\tan\theta=\tan(\theta-\varphi),\ \therefore\ \theta=\frac{\varphi}{2}\ldots\ldots(2).$$

$$\frac{d\theta}{d\varphi}=\tfrac{1}{2}.\quad \frac{ds}{d\varphi}=\tfrac{1}{2}\ .\ 2a\cos\theta\sec(\theta-\varphi)=a\cos\frac{\varphi}{2}\sec\frac{\varphi}{2}=a,$$

$$\frac{ds}{d\varphi}=a,\ S=a\varphi,\quad \text{or } \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}=2a,\ S=2a\theta, \text{but } \theta=\frac{\varphi}{2},\ \therefore S=a\varphi.$$

II. Required the intrinsic equation of the parabola. The polar equation is $r=\dfrac{2a}{1+\cos\theta}=\dfrac{a}{\cos^2\frac{\theta}{2}}$.

$$\frac{dr}{d\theta}=\frac{2a\sin\theta}{(1+\cos\theta)^2}=\frac{a\sin\frac{\theta}{2}}{\cos^3\frac{\theta}{2}}=r\tan(\theta-\varphi)=\frac{a\tan(\theta-\varphi)}{\cos^2\frac{\theta}{2}}.$$

$$\therefore \tan\frac{\theta}{2}=\tan(\theta-\varphi),\quad \therefore\ \theta=2\varphi\ldots\ldots(3).$$

$$\frac{d\theta}{d\varphi}=2.\quad \frac{ds}{d\varphi}=\frac{2a}{\cos^2\frac{\theta}{2}}\sec(\theta-\varphi)=\frac{2a}{\cos^3\varphi}.$$

$$\therefore\ S=\frac{a}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)+\frac{a\sin\varphi}{1-\sin^2\varphi}\quad \text{or thus}$$

$$\frac{ds}{d\theta}=\frac{a}{\cos^3\frac{\theta}{2}}.\quad \therefore\ S=\frac{a}{2}\log\left(\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}\right)+\frac{a\sin\frac{\theta}{2}}{1-\sin^2\frac{\theta}{2}},\ \text{but } \theta=2\varphi.$$

$$\therefore\ S=\frac{a}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)+\frac{a\sin\varphi}{1-\sin^2\varphi}.$$

III. Find the intrinsic equation to the Cardioid. The polar equation is $r=a(1+\cos\theta)=2a\cos^2\frac{\theta}{2}$. $\frac{dr}{d\theta}=-2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$=r\tan(\theta-\varphi)=2a\cos^2\frac{\theta}{2}\tan(\theta-\varphi).$$

$$\therefore\ -\tan\frac{\theta}{2}=\tan(\theta-\varphi),\ \therefore\ 2\varphi=3\theta,\ \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}$$

$$=2a\cos\frac{\theta}{2}, \quad \therefore\ S=4a\sin\frac{\theta}{2}=4a\sin\frac{\varphi}{3}.$$

IV. Find the intrinsic equation to the curve

$$r^{\frac{1}{m}}=a^{\frac{1}{m}}\cos\frac{\theta}{m} \text{ or } r=a\left(\cos\frac{\theta}{m}\right)^m. \quad \frac{dr}{d\theta}=-a\sin\frac{\theta}{m}\left(\cos\frac{\theta}{m}\right)^{m-1}$$

$$=r\tan(\theta-\varphi)=a\left(\cos\frac{\theta}{m}\right)^m\tan(\theta-\varphi).$$

$$\therefore -\tan\frac{\theta}{m}=\tan(\theta-\varphi). \quad \therefore\ (m+1)\theta=m\varphi. \quad \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}$$

$$=a\left\{\left(\cos\frac{\theta}{m}\right)^{2m}+\left(\sin\frac{\theta}{m}\right)^2\left(\cos\frac{\theta}{m}\right)^{2m-2}\right\}^{\frac{1}{2}}=a\left(\cos\frac{\theta}{m}\right)^{m-1}.$$

$$S=a\int\left(\cos\frac{\theta}{m}\right)^{m-1}d\theta=\frac{ma}{m+1}\int\left(\cos\frac{\varphi}{m+1}\right)^{m-1}d\varphi.$$

Let $n=\frac{1}{m}$, then $S=\frac{a}{n+1}\int\left(\cos\frac{n\theta}{n+1}\right)^{\frac{1}{n}-1}d\varphi$ is the intrinsic equation to the curve $r^n=a^n\cos n\theta$.

V. Find the intrinsic equation to the Logarithmic Spiral. The polar equation is $r=be^{\frac{\theta}{c}}$ or $r=ba^{\theta}$, $\log\frac{r}{b}=\frac{\theta}{c}$, $\frac{dr}{d\theta}=\frac{r}{c}=r\tan(\theta-\varphi)$.

$$\therefore\ \tan(\theta-\varphi)=\frac{1}{c}, \quad \theta=\tan^{-1}\frac{1}{c}+\varphi=d+\varphi.$$

$$\frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}=\left(\frac{r^2d\theta^2+dr^2}{d\theta}\right)^{\frac{1}{2}}=\frac{\sqrt{1+c^2}\,dr}{d\theta}.$$

$$\therefore\ ds=\sqrt{1+c^2}\,dr, \quad S=\sqrt{1+c^2}\,.\,r=\sqrt{1+c^2}\,be^{\frac{\theta}{c}}=\sqrt{1+c^2}\,be^{\frac{d+\phi}{c}}.$$

The intrinsic equation for the evolute and involute can be found in the usual way.